Equations

Core Concepts

A An equation is a statement that two expressions are equal.

B The solutions of an equation are the values of the variables that make the resulting numerical statement true.

C The steps in solving an equation are guided by understanding and justified by logical reasoning.

D Equations not solvable in one number system may have solutions in a larger number system.

Core Skills

1. Understand a problem and formulate an equation to solve it.
2. Solve equations in one variable using manipulations guided by the rules of arithmetic and the properties of equality.
3. Rearrange formulas to isolate a quantity of interest.
4. Solve systems of equations.
5. Solve linear inequalities in one variable and graph the solution set on a number line.
6. Graph the solution set of a linear inequality in two variables on the coordinate plane.

Example Tasks

1. **Core Concept A**.
   Are the following equations?
   (a) $y = x^2 + 3x + 2$
   (b) $x^2 + 3x + 2 = 0$
   (c) $x^2 + 3x + 2 = (x + 1)(x + 2)$
   (d) $\left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$
   (e) $x = -2$

2. **Core Concept B**.
   What are the solutions of the equation below?

$$2n(3n - 12) = 0$$

   (a) 0 and 4 (b) 0 and 12
   (c) 2 and 4 (d) 2 and 12

3. **Core Concept B, Core Skill 6**.
   The shaded region inside the triangle ABC is defined by three inequalities. One of these is $x + y < 5\frac{1}{2}$.

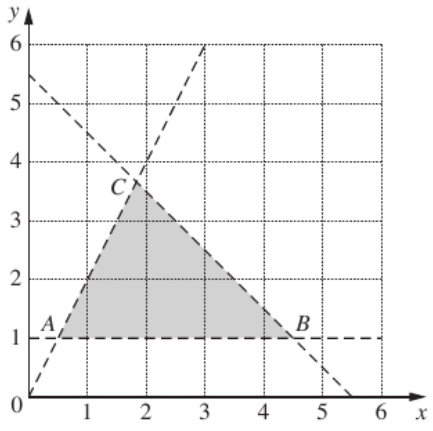

Figure 1

   (a) Write down the other inequalities.
   (b) How many points, with integer coordinates, lie in the shaded region?

4. **Core Concept B**.
   In (a)–(c), does the equation have a solution? Give a reason for your answer that does not depend on solving the equa-

tion.

(a) $\dfrac{t+2}{3+t} = 1$ (b) $\dfrac{3+t}{3-t} = 1$

(c) $\dfrac{t-2}{2-t} = 1$

5. **Core Concept C**.

A student performs the following steps in solving an equation:

$$\frac{x+3}{2x+6} = 1$$
$$x+3 = 2x+6$$
$$x = -3$$

Is the solution correct? If yes, explain why. If no, explain what was wrong with the student's reasoning.

6. **Core Concept C**.

If the equations

$$3x + 2y + 2z = 19$$
$$3x + y + z = 14$$

are true, which of the following is the value of $y + z$?

(a) -5 (b) -4 (c) 0
(d) 4 (e) 5

7. **Core Concept D**.

Write an equation or inequality that has

(a) no real solutions;
(b) infinite numbers of real solutions; and
(c) exactly one real solution.

8. **Core Concept D**.

In (a)–(f), how many solutions are there? Are they rational, real, or complex? Give a reason for your answer that does not depend on solving the equation.

(a) $(x+3)^2 = 9$
(b) $(x-3)^2 = 9$
(c) $-(x-3)^2 = 9$
(d) $16 - (x-3)^2 = 9$
(e) $9(x+3)^2 = 0$
(f) $(x+3)^2 = (x+4)^2$

9. **Core Skill 1**.

One firm offers an investment plan that pays a flat rate of 10% interest each year on the original sum invested. So each dollar grows after n years to

$$(1 + \frac{10n}{100}) \text{ dollars.}$$

Another firm offers a plan that pays 5% interest each year on the previous year's balance. So each dollar grows after n years to

$$(1 + \frac{5}{100})^n \text{ dollars.}$$

Find, using a graphing calculator or a spreadsheet, when the two offers give roughly equal returns. Which is better in the long term?

10. **Equations: Core Skill 1, Core Skill 2; Functions: Core Skill 4**.

Quinn works in Chicago and in New York City. He travels by taxi in each of the two cities.

In Chicago, he pays a fixed taxi fare of \$1.90 per ride, plus \$1.60 per mile traveled.

(a) Write an equation that expresses f, Quinn's total fare for a taxi ride in Chicago, as a function of m, the number of miles traveled.

In New York City, Quinn pays a fixed taxi fare of \$1.50 per ride, plus 25 cents per $\frac{1}{10}$ mile traveled.

(b) Write an equation that expresses f, Quinn's total fare for a taxi ride in New York City, as a function of m, the number of miles traveled.

(c) On a recent trip Quinn noticed that the total number of miles traveled by taxi from the airport to the hotel was the same in each of the two cities. Before tips were added, his taxi fare to the hotel in New York City was \$12.20 more than his taxi fare to the hotel in Chicago. What was the distance from the airport to the hotel in each city? Show or explain how you got your answer.

11. **Core Skill 1, Core Skill 4**.

'Give me 8 sheep and then we will have an equal number' said one shepherd to another. 'No, you give me 8 sheep and then I will have twice as many as you' replied another shepherd. How many sheep did each shepherd have to start with?

12. **Core Skill 1, Core Skill 4**.

(*The Abbot of Canterbury's Puzzle: AD 735–804*)

One hundred bushels of corn were distributed among one hundred people in such a way that each man received three bushels, each woman received two bushels, and each child received half a bushel. Given that there were five times as many women as men, how many children were there?

13. **Core Skill 2**.

Solve the equations

(a) $\frac{24}{x-4} = 1$,
(b) $12 - 2(5 - y) = 5y$.

14. **Core Skill 2**.

Solve the following quadratic equations using factoring methods:

(a) $2x^2+x-3 = 0$ (b) $4x^2 + 6x = 0$
(c) $36x^2 - 25 = 0$ (d) $x^2+6x+9 = 0$

15. **Core Skill 2**.

Use the quadratic formula to solve $4x^2 - 2x = 5$.

16. **Core Skill 3**.

The distance d travelled after time t at a steady speed v is given by

$$d = vt.$$

(a) How long would it take to travel 150 miles at 60 miles per hour?
(b) How fast would you have to go to do it in one and a half hours?

17. **Core Skill 3**.

Solve $A = p + prt$ for p.

18. **Core Skill 3**.

The distance D that it takes a car moving at speed v to stop is given by

$$D = rv + \frac{v^2}{2a}$$

where r is the reaction time it takes the driver to hit the brakes and a is the breaking deceleration.

(a) If r is 2 seconds and $a = 5$ meters/sec^2, how far would the car travel when stopping from 50 meters/sec?
(b) Solve the given equation to find an expression for v in terms of D, r and a. If you want to stop within 100 meters, how fast can you safely go?

COMMENTS ON SOLUTION:

Part (b) goes beyond the scope of Core Skill 3, and illustrates the extension of this task into later course work.

19. **Core Skill 4**.

Solve the simultaneous equations

$$\begin{cases} 2x - y = 16 \\ 3x + 2y = 17. \end{cases}$$

20. **Core Skill 4**.

The only coins that Alexis has are dimes and quarters.

- Her coins have a total value of \$5.80.
- She has a total of 40 coins.

Which of the following systems of equations can be used to find the number of dimes, d, and the number of quarters, q, that Alexis has?

(a)

$$\begin{aligned} d + q &= 5.80 \\ 40d + 40q &= 5.80 \end{aligned}$$

(b)

$$\begin{aligned} d + q &= 40 \\ 5.80d + 5.80q &= 40 \end{aligned}$$

(c)

$$\begin{aligned} d + q &= 5.80 \\ 0.10d + 0.25q &= 40 \end{aligned}$$

(d)

$$\begin{aligned} d + q &= 40 \\ 0.10d + 0.25q &= 5.80 \end{aligned}$$

21. **Core Skill 4**.
Solve the following system of equations:

$$\begin{cases} 2x - y - z = 7 \\ 3x + 5y + z = -10 \\ 4x - 3y + 2z = 4. \end{cases}$$

22. **Core Skill 5**.
Solve $|x - 6| \leq 4$ and locate the solution on the number line.

23. **Core Skill 5**.
Solve $8.5 < 3x + 2 \leq 9.7$ and locate the solution on the number line.

24. **Core Skill 1, Core Skill 5**.
Mr. Smith uses the following formula to calculate students' final score C in his Algebra II class: $C = 0.4E + 0.6T$, where E represents the score on the final exam, and T represents the average score of all tests given during the grading period. All tests and the final exam are worth a maximum of 100 points. The minimum passing score on tests, the final exam, and the course is 60. Determine the inequalities that describe the following situation. When necessary, round scores to the nearest tenth.
(a) Is it possible for a student to have a failing test score average (i.e., $T < 60$ points) and still pass the course?
(b) If you answered "yes," what is the minimum test score average a student can have and still pass the course? What final exam score is needed to pass the course with a minimum test score average?
(c) A student has a particular test score average. How can (s)he figure out the minimum final exam score needed to pass the course?

25. **Core Skill 3**.
If oil should ever be spilled into the Columbia River Estuary, the company responsible for the spill would be liable for monetary damages according to a formula. By Washington state law, the formula in 2009 was given by:[1]

$$D = 0.508GS(A + B + C)$$

In this formula, D is the damage liability in dollars; G is the number of gallons spilled; S is a "vulnerability score" in the range from 1 to 5 that takes into account the wildlife characteristics of any given square kilometer of the estuary[2]; and A, B and C are "chemical penalty scores" in the range from 1 to 5 that take into account the toxicity, harmful mechanical properties, and longevity of the material spilled. For example, kerosene has a toxicity score $A = 1.4$, a harmful mechanical property score $B = 2.4$, and a longevity score $C = 1$.[3] Suppose that a company responsible for a kerosene spill in an area of lowest vulnerability is held liable for \$10 million. How many gallons were spilled? How many dollars per gallon was the company charged for the spill? In general, what is a formula for the number of dollars of liability per gallon of spill? What is the maximum possible liability in dollars per gallon?

Sources

1. McCallum, William
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[1] http://apps.leg.wa.gov/WAC/default.aspx?cite=173-183-840
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[3] http://apps.leg.wa.gov/WAC/default.aspx?dispo=true&cite=173-183&full=true.

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